Optimal Contracts with Exogenous Risk

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Abstract. I study the relation between the level of exogenous "risk" and pay-for-performance sensitivity (PPS) of optimal contracts. I first show that none of the known restrictions allow for exogenous risk in the standard principal-agent model. Next, I identify restrictions that support using the first-order approach for sufficiently high levels of exogenous risk. Third, by placing further restrictions on the agent's preferences I provide a parametric example where there is not a monotone relation between risk and PPS. Finally, by appealing to the limiting case of this example I make the case that optimal contracts are unlikely to exhibit a monotone relation between risk and PPS when risk is exogenous.

1. Introduction

The study of optimal contracts in principal-agent relations rests to a large extent on the ability to formulate the contracting problem in a tractable manner. Unfortunately, while the first-order approach offers the tractability needed, applying it comes at a cost. As is well known since (and due to) Mirrlees [1975], the approach is not generally valid and thus neither are the results the approach facilitates. As first proposed by Mirrlees [1977], however, and subsequently proven by Rogerson [1985], validity of the first-order approach can be assured if attention is confined to production functions satisfying the so-called MLR and CDF conditions. Roughly, the former suggest that higher outcomes be more consistent with higher effort, while the latter corresponds to a form of stochastically diminishing return to effort in the production function.

While from a theoretical perspective these conditions have (some) intuitive appeal, from a more practical point of view these conditions are less satisfying as few examples of production functions that have these properties have been identified.[1] For example, the continuous equivalent of the discrete distribution with these properties provided in Rogerson [1985], while possessing the same desired properties, does not lend itself

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[1] Recent noteworthy exceptions are the two groups of distributions introduced by LiCalzi and Spaeter [2003].

to the first-order approach as it is easily shown to suffer form the Mirrlees [1975] non-existence problem.[2]

Jewitt [1988] takes a different approach to extending the reach of the first-order approach. By placing arguably somewhat weak additional restrictions on the contracting parties preferences, along with a slightly stronger condition on the likelihood ratio (concavely increasing) he provide access to a broader class of productions functions that satisfy (also) arguably somewhat less stringent conditions than the CDF condition.[3] These distributions are all bounded from below and seem to provide reasonable representations of phenomena of significant interest in contract theory (and practice) such as the (empirical) distribution of stock prices. Despite these efforts to map out new territory conquerable by the first-order approach, however, it still seems fair to suggest that the current territories occupied are small at best and quite constraining realistically.

Seemingly unfazed by the practical challenges involved in characterizing optimal incentive contracts in standard models of moral hazard, a large body of empirical work has emerged testing a central prediction attributed to the principal-agent theory: an inverse relation between "risk" and "pay-for-performance sensitivity" (PPS hereafter). This relation, while often in the empirical literature illustrated using the contractual form of Holmstrom and Milgrom [1987], is typically ascribed to principal-agent theory in general without having encountered much of a theoretical challenge. In (fitting) contrast high profile empirical studies such as those by Jensen and Murphy [1990], Haubrich [1994], Garen [1994], Aggarwal and Samwick [1999], and Core and Guay [2002] have provided only limited or, perhaps more appropriately characterized, contradictory evidence on this relation.[4]

The insistence on an inverse equilibrium-relation between risk and the sensitivity of pay to performance in optimal contracts targeted at the standard problem of moral hazard appears based on a view of risk being exogenous. More specifically, production functions where the agent's choice is confined to selecting only the mean of a symmetric, unimodal, continuous output distribution to allow for the exogenously determined dispersion to serve as a measure of "risk." For such models the logical equilibrium link between risk and incentive then seems straight forward to establish. Presumably, if the dispersion of the outcome distribution increases, a (risk-neutral) principal would respond by decreasing the variation in the sharing rule to attain the optimal trade-off between improving risk-sharing and weakening of incentives.

[2] Specifically, Stole (2001) suggests that a continuous version of the distribution of the distribution provided by Rogerson (1985), a form of Beta-distribution, satisfies the *MLR* and *CDF* conditions. While the particular distribution he suggests does not satisfy the CDF condition, it is easily verified that for Beta versions that do, f^a/f falls without limit when x approaches its lower limit.

[3] The MLR condition remains central though.

[4] See also Prendergast [1999] and Chiappori and Salanie [2000] for overviews on this literature.

While this link is intuitively appealing the simple fact is, however, that little is known about the nature of optimal contracts in production environments of this type. Indeed, due to methodological constraints, no results are available that link exogenous output risk to the PPS of optimal contracts for standard basic (one shot) principal-agent models. The purpose of this paper is to fill out methodological gaps that have stood in the way of doing this. Then, in turn, to use the methodological advances to achieve insights into the (equilibrium) "risk/PPS" trade-off predicted by the standard principal-agent model.

Several specific issues are addressed to this end. First, I establish that using the first-order approach to characterize the optimal risk/incentive trade-off for production functions of the "effort-plus-noise" type cannot be supported by neither the conditions provided by Mirrlees and Rogerson, nor by those provided by Jewitt [1988].[5] Second, I then demonstrate that a necessary condition for the first-order approach to be valid for such production functions is that the dispersion of the noise-term (the risk) is "sufficiently large." This makes the comparative static exercise relating risk to the PPS of sharing rules derived using the first-order approach somewhat fragile. Finally, however, I do demonstrate in a setting where the first-order approach *is* valid that the PPS of the optimal sharing rule is not monotonically decreasing in the dispersion of the noise term!

2. The Basic Model

Given the issues at hand are optimal sharing rules for production functions where risk is entirely exogenous and, in particular, the ability of the first-order approach to deliver such, the production functions to be considered here are on the form

$$x = a + \epsilon(\theta), \tag{1}$$

where $a \in \mathcal{R}^{+}$ is the agent's "effort," and $\epsilon(\theta)$ is the random (or risky) component parameterized by θ - the determinant of the dispersion of the random component. Here, lower values of θ is taken to imply that $\epsilon(\theta)$ is less risky in the sense of second-order stochastic dominance. I use $f(x, a, \theta)$ to denote the density of x for given values of a and θ while $F(x, a, \theta)$ is used to represent the corresponding distribution function.

For simplicity and consistent the literature relating risk to PPS , the principal here is taken to be risk-neutral while the risk-averse and effort-averse agent has (standard) additively separable preferences, $U(s(x), a) \equiv u(s(x)) - v(a)$ where $s\,(x)$ is the agent's share of realized output, $u(s(x))$ is the agent's increasing and strictly concave utility for income defined over the entire real line, and $v(a)$ is the agent's strictly increasing, convex, and twice continuously differentiable cost of effort. $h(u(s(x))) \equiv s(x)$ is the

[5] Specifically I show that no production function of this type exists that satisfies either the Mirrlees/Rogerson or the Jewitt conditions for which the first-order approach is valid.

inverse of the agent's utility of consumption. The agent's reservation utility is here represented by $\underline{U}$ and the timing of events is the usual.

For an agency of this form the following observation (adapted from Jewitt [1988]) is of central importance:

Observation 1. *When the production function is of the form (1), then* $F(x, a, \theta) = F(x - a, \theta)$.

Indeed, this observation forms the basis for the first result:

Proposition 1. *No production function of the form (1) satisfying either* (i) *the Jewitt-conditions or* (ii) *the Mirrlees-Rogerson-conditions exists for which the first-order approach (FOA) to characterizing the optimal solution to the principal's problem is valid.*

Proof of Proposition 1. First, note that when $f(x, a, \theta) = f(x - a, \theta)$, the support of x must be unbounded from below to avoid moving support and, thus, first-best. To prove part (i) then, suppose that the FOA is valid so that the optimal contract satisfies

$$\frac{1}{u'(s(x))} = \lambda + \mu \frac{f^a(x, a, \theta)}{f(x, a, \theta)}, \tag{2}$$

where λ and μ are the Lagrange multipliers associated with the (IR) and the (IC) constraints respectively. With the support of x unbounded from below, condition (2.11) of Jewitt [1988] implies that $f^a(x, a, \theta)/f(x, a, \theta) \to -\infty$ as $x \to -\infty$. Since validity of the FOA implies that $\mu > 0$,[6] the RHS of (2) falls without limit as $x \to -\infty$. But the LHS of (2) is always strictly positive contradicting the presumption that the FOA is valid. This concludes the first part of the proof.

To prove part (ii) note that $F^{aa}(x-a) = f^x(x-a)$. $CDFC$ thus requires $f^x(x-a)$ to be everywhere non-negative and strictly positive for some x.[7] Since the support of x must be unbounded from below and since $\int_{-\infty}^{\overline{\epsilon}} f(\epsilon)dx = 1$, where $\overline{\epsilon}$ is the upper bound of the support of $f(\epsilon)$, $CDFC$ implies that $\overline{\epsilon}$ must be finite and that $f(\overline{\epsilon}) > 0$. Accordingly, then

$$f^a(x - a) = \begin{cases} -f^x(x - a), & for\ x < \overline{\epsilon} + a \\ f(\overline{\epsilon}) > 0, & for\ x = \overline{\epsilon} + a \end{cases},$$

and the sharing rule characterized using the FOA is thus increasing discontinuously at $x = \overline{\epsilon} + a^*$. Then, since $v(a)$ is strictly increasing, twice continuously differentiable, EU must be decreasing in a as a approaches a^*. Accordingly, a^* is not a global maximum in the agent's choice problem. $Q.E.D.$

[6] Holmstrom (1979) Proposition 1. Also, Jewitt (1988) established *without* first assuming that the FOA is valid that in the case of a risk-neutral principal, $\mu > 0$ if a is interior.

[7] Jewitt (1988).

3. Alternative Restrictions

Given the incompatibility of production functions of the form (1) with the standard restrictions used to ensure the validity of the FOA, any insight into the relation between exogenous risk and PPS requires identification of alternate restrictions to be placed on the production function as well as on the agent's preferences. At the outset it seems quite unlikely that one would be able to find such restrictions unless the sharing rule is monotone non-decreasing in x. As a starting point it therefore seems natural to confine attention to production functions that satisfy the MLR condition.[8] This immediately places restrictions on the general shape of $f(x, a, \theta)$.

Lemma 1. *When the production function is of the form (1), a necessary condition for the MLR condition to be satisfied is that $f(x, a, \theta)$ is unimodal.*

Proof of Lemma 1 Suppose $f(x, a, \theta)$ is not unimodal. Then, for any given a and θ there exists corresponding values of x, say $\underline{x}$ and $\overline{x}$, $\underline{x} < \overline{x}$, such that $f^x(x, a, \theta)|_{x=\underline{x}} < 0$ and $f^x(x, a, \theta)|_{x=\overline{x}} > 0$. Since $f^x(x, a, \theta) = -f^a(x, a, \theta)$ when the production function is of the form (1), $f^a(\underline{x}, a, \theta)/f(x, a, \theta) > f^a(\overline{x}, a, \theta)/f(x, a, \theta).Q.E.D.$

For the purpose of concreteness and to facilitate further analysis I confine my attention to a class of unimodal distributions that are also symmetric. Specifically:

$$f(x, a, \theta) = g\left(\frac{-h(|x-a|)}{\theta}\right) k(\theta), \tag{3}$$

where $g' > 0$, $g(0) > 0$ and finite, $\lim\limits_{\frac{h(|x-a|)}{\theta} \to \infty} g\left(\frac{-h(|x-a|)}{\theta}\right) \to 0$, $h(0) = 0$, $h' > 0 \, \forall x \neq a$, and $k(\theta) > 0$ satisfies $d \int_{-\infty}^{\infty} g\left(\frac{-h(|x-a|)}{\theta}\right) k(\theta) dx/d\theta = 0$. Besides arguably being a natural choice, confining attention to symmetric distributions is motivated by two factors. First, comparing the riskiness of skewed distributions is more involved and places restrictions on the type of skewness that can be considered anyway. Second, the empirical work on the relation between risk and PPS typically make reference to the variance of a symmetric distribution as the proper measure of risk.

That the likelihood ratio must be monotone and, from the proof of Proposition 1, is required to be bounded then places further restrictions on h and g:

Lemma 2. *For the class of production functions given by (3), the likelihood ratio is monotonic non-decreasing and bounded if and only if $h'g'/g$ is monotonic non-decreasing in x and $f(x, a, \theta) \to k(\theta)Exp[-B|x-a|/\theta + R]$* as $|x-a|/\theta \to \infty$.

Proof of Lemma 2. With (3) we have

$$f^a = Sign[x-a]\frac{h'}{\theta} g' k(\theta),$$

[8] It follows directly from (1) that when the first-order approach is valid, $s(x)$ is monotone non-decreasing in x if and only if f^a/f is monotone non-decreasing in x.

so that

$$\frac{f^a}{f} = Sign[x-a]\frac{h'}{\theta}\frac{g'}{g}, \tag{4}$$

which is monotonic non-decreasing in x if and only if $h'g'/g$ is. Moreover, given (4) $\lim_{(|x-a|/\theta)\to\infty} \frac{f^a}{f} = B > 0$ implies $\lim_{(|x-a|/\theta)\to\infty} \frac{g'}{g} = \frac{B}{h'}$ or that $\frac{1}{g}dg = Bdy$. Solving yields the second requirement in Lemma 2.$Q.E.D.$

Corollary 1. *For the class of production functions given by (3) with likelihood ratios bounded above at* $B > 0$, $\lim_{(|x-a|/\theta)\to\infty} h'(\cdot) \to B$.

Lemma 3

$$\lim_{\theta\to 0}\frac{d(f^a/f)}{dx} = \begin{cases} 0, & for\ x \neq a \\ \infty, & for\ x = a \end{cases} \tag{5}$$

Proof of Lemma 3 Since $\frac{d(f^a/f)}{dx} = \frac{h''}{\theta} - \left(\frac{h'}{\theta}\right)^2\left[\frac{g''}{g} - \left(\frac{g'}{g}\right)^2\right]$, Lemma 2 yields the first line. With $\frac{d(f^a/f)}{d\theta}|_{(|x-a|/\theta)=0} = -\frac{h'}{\theta^2}$ monotonicity of f^a/f yields the second line. $Q.E.D.$

While the necessary restrictions identified above unfortunately rule out many of the best known standard distributions that could be useful as a platform for further analysis, the class of candidate distributions defined by (3) and Lemma 2 actually does contain (at least) one standard distribution. Specifically the (Double Exponential) Laplace distribution:

$$f(x,a,\theta) = \frac{1}{2\theta}e^{\frac{-|x-a|}{\theta}}, \tag{6}$$

which correspond to the special case of $g(\cdot) = Exp(\cdot)$, $h(|x-a|) = |x-a|$, and $k(\theta) = 1/2\theta$. Although perhaps somewhat unusual from an empirical vantage point, the specific properties of this distribution are quite instructive and useful for identifying the general fallacy in asserting that the amount of exogenous risk must be inverse related to the PPS in optimal incentive contracts.[9] For now consider therefore the case where the production function is of the form (6) so that the agent's effort a is the mean of a Laplace distribution with dispersion θ.

Being an extreme case of the class of candidate distributions defined by (3) and Lemma 2, this distribution does have somewhat extreme properties. In particular, for this distribution,

$$f^a(x,a,\theta) = \frac{Sign[x-a]}{\theta} \times f(x,a,\theta),$$

and thus

$$\frac{f^a(x,a,\theta)}{f(x,a,\theta)} = \frac{Sign[x-a]}{\theta},$$

[9] I will return to this in the Discussion section.

which is constant for $x \neq a$. This property makes the relation of the risk-PPS trade-off particularly easy to examine because the optimal contract when the FOA is valid simply provides the agent the following levels of utility

$$u(s(x)) = \begin{cases} \underline{U} + v(a^\theta) + \theta v'(a^\theta), & for\ x \geq a^\theta \\ \underline{U} + v(a^\theta) - \theta v'(a^\theta), & for\ x < a^\theta \end{cases}, \tag{7}$$

independent of the relation between u and $1/u'$![10] The independence between $s(x)$ and the properties of u follows since for this distribution, $[\lambda + \mu \frac{f^a}{f}]$ takes on only two different values and both the (IR) and the (IC) constraint still must be (exactly) satisfied.

With this representation of the optimal contract, the PPS is measured unambiguously by the difference in the contractual payments between the $x < a^\theta$ and the $x \geq a^\theta$ scenarios. The standard empirical hypothesis that increased risk here measured by the Laplace distribution's standard deviation, θ, should be accompanied by a decrease in the PPS in the agent's contract, for this environment therefore is a statement about the equilibrium relation between θ and $h(\underline{U}+v(a^\theta)+\theta v'(a^\theta)) - h(\underline{U}+v(a^\theta)-\theta v'(a^\theta))$, the difference in the payment to the agent when output exceeds and falls short of expectations. The remainder of this section explores this relation and in doing so demonstrates the lack of any particular directional relation between risk and PPS.

Since the PPS in the optimal contract for the Laplace-case is monotonic increasing in:

$$\theta v'(a^\theta), \tag{8}$$

it would seem to be a straight forward task to conclude that the relation between exogenous risk and PPS cannot be everywhere decreasing. This appears to follow because when $\theta \to 0$, presumably $a \to a^{FB}$ and with $v'(a^{FB}) \leq 1$ for the production function here specified, $\theta v'(a^\theta) \to 0$ from above as $\theta \to 0$.[11]

The problem with the above argument is closely related to Lemma 2. As $\theta \to 0$, $\partial EU/\partial a \to -v'(a)$ for $a < a^{FB}$. Moreover, as $s(x) \to \underline{U} + v(a^{FB})$, $EU(s(x), 0) \to \underline{U} + v(a^{FB})$ as well. As under the optimal contract $EU(s(x), a^\theta) = \underline{U}$ independent of θ, in the limiting case of $\theta = 0$, $a = 0$ then is the agent's optimal response to the contract (7) implying that the first-order approach is not valid in this case. Accordingly, establishing that the relation between risk and PPS is not always everywhere decreasing by appealing to (8) requires that it be established that preferences exist such that the first-order approach is valid for a range of values of θ preferably close to 0. Proposition 2 first addresses the possibility that the FOA is valid for any given value of $\theta > 0$.

[10] a^θ denotes the second-best value of a for a given θ.

[11] a^{FB} is here used to signify the first-best effort level.

Proposition 2. *When the production function is of the form (6) the FOA is valid at θ if $v(a)$* satisfies $v'(a^\theta)e^{\frac{a-a^\theta}{\theta}} - v'(a) > 0$, $a < a^\theta$.
Proof of Proposition 2. A sufficient condition in the case of (6) for the first-order approach to be valid is that $\partial EU(s(x), a)/\partial a > 0$ for $a < a^\theta$, where $s(x)$ is the contract (7) derived using the first-order approach. This follows since by construction, $\partial EU(s(x), a)/\partial a = 0$ for $a = a^\theta$ and, as is easily verified, in the Laplace case $\partial EU(s(x), a)/\partial a < 0$ for $a > a^\theta$. With the structure of this contract,

$$EU(s(x), a) = \underline{U} + v(a^\theta) - \int_{-\infty}^{a^\theta} \theta v'(a^\theta) f(x, a, \theta) dx + \int_{a^\theta}^{\infty} \theta v'(a^\theta) f(x, a, \theta) dx - v(a),$$

so that

$$\partial EU(s(x), a)/\partial a = 2\theta v'(a^\theta) f(a^\theta, a, \theta) - v'(a).$$

Accordingly, the first-order approach is valid if $v(a)$ satisfies

$$v'(a) < 2\theta v'(a^\theta) f(a^\theta, a, \theta), \qquad a < a^\theta,$$

or, using (6),

$$v'(a^\theta)e^{\frac{a-a^\theta}{\theta}} - v'(a) > 0, \qquad a < a^\theta.$$

As $v'(a^\theta) > 0$, $e^{\frac{-a^\theta}{\theta}} > 0$, and $e^{\frac{a-a^\theta}{\theta}}$ is increasing convexly in a for $a < a^\theta$, it is always possible to find an increasing and convex cost function $v(a)$ with $v(0)$ that satisfies this condition. $Q.E.D.$

Proposition 2 simply establishes that in the Laplace case, given a $\theta > 0$ and a solution a^θ derived using the FOA, this is indeed the optimal solution to the principal's problem as long as the agent's cost function does not increase "too quickly" in a for $a < a^\theta$. Proposition 3 establishes the implications of this constraint for the relation between θ and a^θ.
Proposition 3. *When the production function is of the form (6) and $v(a)$ satisfies $v'(a^\theta)e^{\frac{a-a^\theta}{\theta}} - v'(a) > 0$, $a < a^\theta$, where a^θ is obtained using the FOA, then $\frac{da^\theta}{d\theta} < 0$.*
Proof of Proposition 3. Given the nature of the optimal contract in the Laplace case as given by (7), the principal's residual maximization problem can be expressed as

$$\max_a \quad a - \frac{1}{2}h[\underline{U} + v(a) - \theta v'(a)] - \frac{1}{2}h[\underline{U} + v(a) + \theta v'(a)]$$

which has the first-order condition (suppressing (a) from the notation for the moment)

$$FOC \equiv 2 - (v' - \theta v'')h'[\underline{U} + v - \theta v'] - (v' + \theta v'')h'[\underline{U} + v + \theta v'] = 0.$$

Furthermore, taking the partial derivatives of the left-hand-side of this first-order condition with respect to θ and a respectively yield

$$\frac{\partial FOC}{\partial \theta} = v''h'[L] + v'(v' - \theta v'')h''[L] - v''h'[H] - v'(v' + \theta v'')h''[H], \tag{9}$$

and

$$\frac{\partial FOC}{\partial a} = -(v''-\theta v''')h'[L]-(v'-\theta v'')^2 h'[L]-(v''+\theta v''')h'[H]-(v'+\theta v'')^2 h''[H]), \tag{10}$$

where $L \equiv \underline{U} + v(a^\theta) - \theta v'(a^\theta)$, and $H \equiv \underline{U} + v(a^\theta) + \theta v'(a^\theta)$. Accordingly,

$$\frac{da^\theta}{d\theta} = -\frac{v''h'[L] + v'(v' - \theta v'')h''[L] - v''h'[H] - v'(v' + \theta v'')h''[H]}{-(v'' - \theta v''')h'[L] - (v' - \theta v'')^2 h''[L] - (v'' + \theta v''')h'[H] - (v' + \theta v'')^2 h''[H]}.$$

where the denominator is the second-order condition for a^θ and thus guaranteed to be negative and the sign of $\frac{da^\theta}{d\theta}$ is thus determined by the sign of the numerator.

Now, the requirement that $v'(a^\theta)e^{\frac{a-a^\theta}{\theta}} - v'(a) > 0$, $a < a^\theta$ can be re-written as $v'(a^\theta)e^{\frac{a-a^\theta}{\theta}} - \delta(a,\theta) = v'(a)$, where $\delta(a,\theta) > (=)\ 0$ for $a < (=)\ a^\theta$. Accordingly, $\delta'(a^\theta,\theta) \leq 0$. Now differentiate $v'(a)$ to get $\theta v''(a) = v'(a^\theta)e^{\frac{a-a^\theta}{\theta}} - \delta'(a,\theta)$. Accordingly, the requirement that $v'(a^\theta)e^{\frac{a-a^\theta}{\theta}} - v'(a) > 0$ implies that $\theta v''(a^\theta) \geq v'(a^\theta)$. This then confirms that the numerator of the expression for $\frac{da^\theta}{d\theta}$ is also negative. *Q.E.D.*

Using the relation between a and θ from proposition 3, proposition 4 now expands on proposition 2 to establish that if the agent's cost function supports using the FOA for a given $\widehat{\theta} > 0$, the FOA is also valid for all $\theta > \widehat{\theta}$.

Proposition 4. *When the production function is of the form (6) and $v(a)$ satisfies $v'(a^{\widehat{\theta}})e^{\frac{a-a^{\widehat{\theta}}}{\widehat{\theta}}} - v'(a) > 0$, $a < a^\theta$, where a^θ is obtained using the FOA, then the FOA is valid for all $\theta \geq \widehat{\theta}$.*

Proof of Proposition 4. Consider some strictly positive value of θ, say $\widehat{\theta}$, and define θ_ε as the value of θ for which $a^{\theta_\varepsilon} = a^{\widehat{\theta}} - \varepsilon$, $\varepsilon \geq 0$. Then, if a^θ is decreasing in θ, $\theta_\varepsilon \geq \widehat{\theta}$. I can then re-write the condition from proposition 2 as

$$e^{\frac{-|a^{\widehat{\theta}}-\varepsilon-a|}{\theta_\varepsilon}} - \frac{v'(a)}{v'(a^{\theta_\varepsilon})} > 0, \qquad a < a^{\theta_\varepsilon} \text{ and } \varepsilon = 0. \tag{11}$$

Now differentiate the LHS of (11) with respect to ε to obtain

$$\left(\frac{1}{\theta_\varepsilon} + \frac{d\theta_\varepsilon}{d\varepsilon}\frac{(a^{\widehat{\theta}} - \varepsilon - a)}{\theta_\varepsilon^2}\right) e^{\frac{-|a^{\widehat{\theta}}-\varepsilon-a|}{\theta_\varepsilon}} + \frac{v'(a)}{v'(a^{\theta_\varepsilon})}\frac{v''(a^{\theta_\varepsilon})}{v'(a^{\theta_\varepsilon})} > 0, \qquad \forall a \leq a^{\widehat{\theta}} - \varepsilon > 0.$$

Accordingly, if the first-order approach is valid for $\theta = \widehat{\theta}$, it is valid for all $\theta \in [\widehat{\theta}, \infty)$. *Q.E.D.*

4. The "Risk - *PPS*" Trade-off

While the preceding analysis establishes that it is indeed feasible to place specific restrictions on preferences and the production function to guarantee the validity of the FOA, the task of establishing generally a relation between risk and PPS does not appear within reach even with the benefits offered by the Laplace representation. The reason is that the validity of the FOA is crucially dependent on the risk in the production function in that for any given set of preferences the validity of the FOA requires that the risk is not "too low." In this section I therefore restrict the functional forms further to facilitate a specific (counter-) example to demonstrate that the maintained proposition that optimal contracts exhibit an inverse relation between PPS and exogenous risk is false.

To that end I utilize the following representation of the agent's preferences:

$$U(s(x), a) = (s(x))^{\frac{1}{2}} - a^2/2,$$

and set $\underline{U} = 1$. With this representation, under the maintained assumption that the FOA is valid, the optimal level of effort, a^{θ}, is simply the solution to the following polynomial:

$$1 - 2a - a^3 - 2\theta^2 a = 0.$$

Table 1 below summarizes the solution along with the properties of the contract derived using the FOA as well as indicates the actual validity of the FOA at the various levels of risk as per the condition in proposition 2.

θ	a^θ	$2\theta v'(a^\theta)$	$s(H) - s(L)$	FOA
.1	0.44995	0.08999	0.19820	-
.2	0.43986	0.17594	0.38593	-
.3	0.42380	0.25428	0.55423	-
.4	0.40285	0.32228	0.69686	-
.5	0.37834	0.37834	0.81084	+
.6	0.35166	0.42199	0.89617	+
.7	0.32414	0.45380	0.95527	+
.8	0.29690	0.47504	0.99196	+
.9	0.27076	0.48737	1.01047	+
1.0	0.24627	0.49254	1.01495	+
1.1	0.22371	0.49216	1.00896	+
1.2	0.20230	0.48552	0.99091	+
1.3	0.18470	0.48022	0.97682	+
1.4	0.16812	0.47074	0.95477	+
1.5	0.15329	0.45987	0.93055	+
1.6	0.14006	0.44819	0.90518	+
1.7	0.12826	0.43608	0.87934	+
1.8	0.11773	0.42383	0.85353	+
1.9	0.10832	0.41162	0.82806	+
2.0	0.09990	0.39960	0.80319	+

The column to the right indicates if the FOA is (+) or is not (-) valid for the given level of θ. The third and fourth columns of this table provides the PPS in the contract respectively in terms of utiles and consumption units. The latter's relation with risk as measured by θ is plotted in figure1 below.

Insert Figure 1 about here.

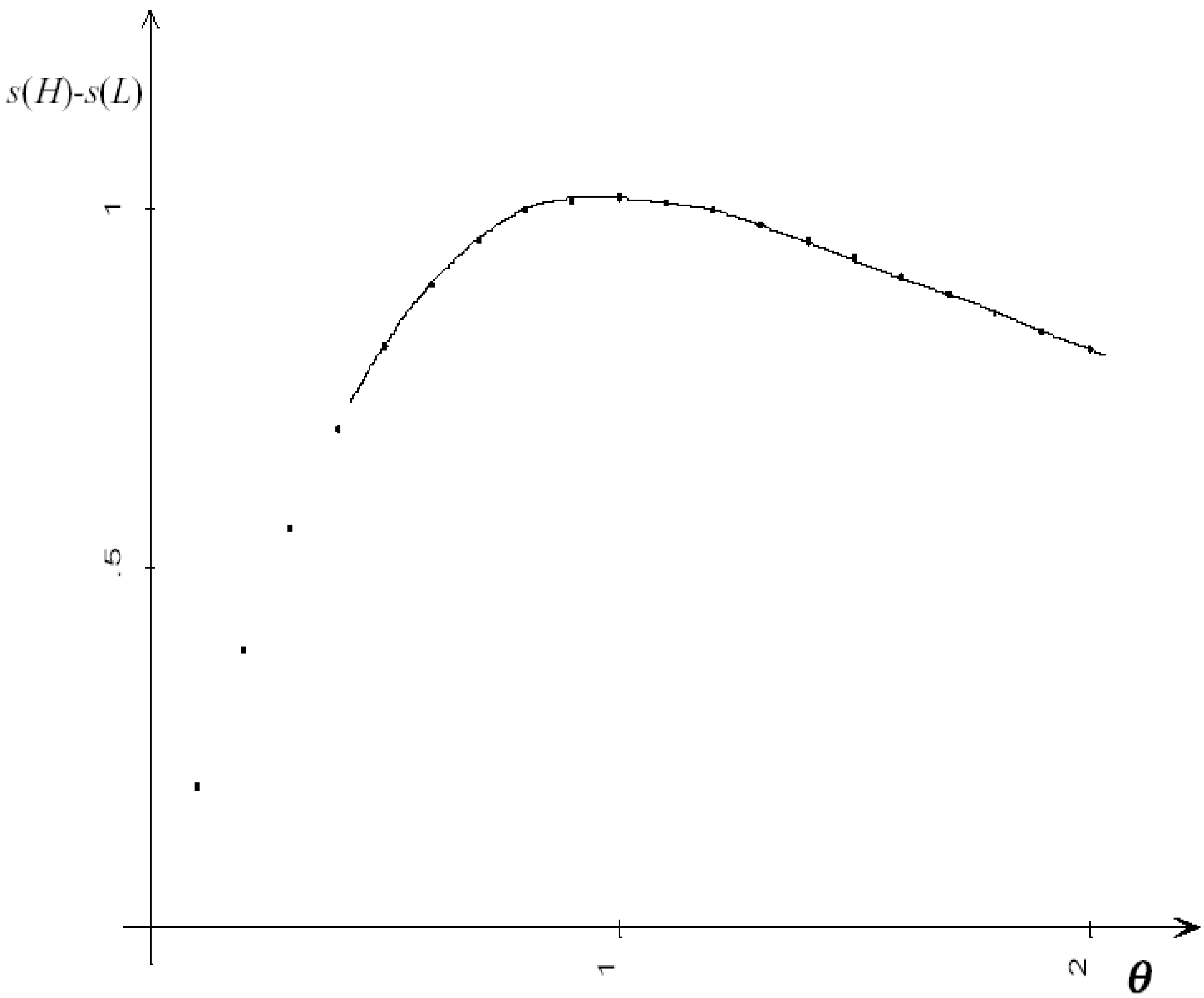
The solid curve in this figure represents the relation between risk and PPS over the range of θ where the FOA *is* indeed valid and the PPS listed in the table therefore is the PPS of the optimal contract. As can be seen, the validity of the FOA hinges on the value of θ. For the particular preference representation the FOA attains validity for a value of θ between .4 and .5 and is, as per proposition 4, (in-) valid for all (smaller) greater values of θ.

What is most important about this example is, however, that it establishes that the PPS is not monotonically decreasing in θ over the range where the FOA is actually valid. This is only the case for relatively large values of θ while the relation between risk and PPS is actually reversed for relatively small values of θ. Unless the level of risk somehow is expected to always be "large" there is no theoretical reason for expecting that empirically, higher risk leads to lower PPS. Indeed, the theory predicts mixed and/or weak empirical results unless, perhaps, the sample is partitioned on risk.

5. Discussion and Conclusion

The analysis in this paper provides several points. First, it is possible to place restrictions on preferences and production functions in such a way that the optimal sharing rule can be derived and studied for the standard principal agent model with exogenous risk. Second, that there is no theoretical support for the claim that the standard principal-agent model predicts an inverse relation between exogenous risk and PPS. In the specific case studied it was shown that when risk is sufficiently low, a positive relation exists. Third, and somewhat depressing, finding general restrictions to validate this type of analysis is unlikely. This follows since the restrictions needed on the agent's preferences depend on the nature of the optimal contract and except in the case of the Laplace distribution where the functional form of the optimal contract is extremely simple, identifying even sufficient restrictions is not likely to be doable.

Despite the lack of restrictions that would allow for a more general analysis of optimal contracts in settings with exogenous risk, however, the nature of the optimal relation between PPS and such exogenous risk is bound to be the same as in the special case analyzed here. This follows from the fact that any production function must satisfy lemma 2, implying that the optimal contracts are bounded from above as well as from below. As in the limit when the risk goes to zero, all contracts and all distributions "look" the same, the PPS must be approaching 0 as in the case presented here. Similarly, as risk grows without limit no effort will be elicited and the optimal PPS then should be approaching 0 as well. Thus, while whether the optimal contracts can actually be derived in alternative settings remains an open question, the analysis here strongly suggests the if settings exist where this is feasible, the contracts will also exhibit a non-monotone relation between risk and PPS.


s(H)-s(L)
1
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